Solving an Operator Equation by Iteration

P. G. Spain
University of Glasgow
Glasgow, Scotland

Submitted by Hans Schneider

ABSTRACT

It was shown by Bushell [1] that the equation $t'xt = x^2$ has a unique positive-definite solution when t is a real invertible matrix; the proof utilizes the Hilbert projective metric and the Banach fixed-point theorem. I present a simpler proof of a more general result.

Let H be a Hilbert space (of any dimension) and $\mathbf{I}$ the set of positive operators on H having norm no greater than 1.

THEOREM. *Let $f:\mathbf{I}\to\mathbf{I}$ be an increasing function that is continuous in the strong operator topology and let $a \leqslant f(a)$ for some a in $\mathbf{I}$. Then the sequence $a, f(a), f(f(a)),\ldots$ increases to a solution of the equation $f(x)=x$.*

Proof. The sequence $a, f(a), f(f(a)),\ldots$ increases in $\mathbf{I}$ and therefore has a supremum, say x, to which it converges in the strong operator topology. Since f is continuous in this topology, we have $f(x)=x$. ■

COROLLARY. *Let t be an invertible operator on H and p a real number, $0<p<1$. Then the equation $t^*x^pt=x$ has a positive non-zero solution. Moreover, this equation has only one positive invertible solution.*

Proof. There is no loss of generality in assuming that the norm of t is no greater than 1. Then, defining $f:\mathbf{I}\to\mathbf{I}:x\mapsto t^*x^pt$, we see that f increases and is continuous in the strong operator topology.

Let $r=\min \operatorname{sp}(t^*t)$ (>0), $s=r^{1/(1-p)}$ and $a=s\mathbf{1}$, where $\mathbf{1}$ is the identity operator on H. Then $\max \operatorname{sp}(a)=s=s^pr=\min \operatorname{sp} f(a)$; so $a\leqslant f(a)$. The sequence $a, f(a), f(f(a)),\ldots$ therefore increases to a solution of the equation $f(x)=x$. This solution is invertible, for a is invertible.

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If y is another positive invertible operator and $f(y)=y$, then there is a positive real number q for which $x \leqslant qy$. Then $t^* x^p t \leqslant q^p t^* y^p t$; so $x \leqslant q^p y$. It follows that $x \leqslant q^{p^2} y$, $x \leqslant q^{p^3} y, \ldots$; so $x \leqslant y$. Similarly, $y \leqslant x$. ∎

I am grateful to Dr. N. J. Young for bringing [1] *to my attention.*

REFERENCES

1 P. J. Bushell, On solutions of the matrix equation $T'AT = A^2$, *Linear Algebra Appl.* 8 (1974), 465–469.

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